On Operads

J.M. Davies

Abstract

A operad is essentially an algebraic book keeping tool. Developed out of a need to systematically document complicated associativity and commutivity relations, they now find many uses in algebraic topology, homotopical algebra, and even physics. Since we are interested in $\infty$-categories and types of homotopical algebras, then we are interested in the topological origins of operads, and their use to track higher homotopy coherence in the study of higher category theory and higher algebra.

Contents

1 Operads and Algebras 1
2 Fundamental Examples 4
3 Operads in Topology 7
4 Operads in Algebra 9
5 Coloured Operads 9

Setup

Let $\mathcal{C}$ be a symmetric monoidal category with finite coproducts, with product $\otimes$ and unit $1$. We will often also demand that the tensor product distributes over the coproduct, so

$$(A \sqcup B) \otimes C = (A \otimes C) \sqcup (B \otimes C),$$

and that $\mathcal{C}$ is closed, so it has an internal hom and the following adjunction,

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, \text{hom}(B, C)).$$

When we write $A^n$ for some $A \in \mathcal{C}$, then we mean the $n$-fold tensor product of $A$ when $n \geq 0$, and $A^0$ is simply $1$.

In such a closed symmetric monoidal category $\mathcal{C}$, for any given finite set $K$ and any object $A \in \mathcal{C}$, we define $A[K]$ to be the coproduct of $A$ with itself, indexed over the elements of $K$.

1 Operads and Algebras

Our first item of business is the define an operad $\mathcal{P}$ in our category $\mathcal{C}$, using the original definition of J.P. May.
Definition 1.1 (Operad). An operad $\mathcal{P}$ in a category $\mathcal{C}$ is a collection of objects $\mathcal{P}(j)$ for each $j \geq 0$, where $\mathcal{P}(j)$ has a right $\Sigma_j$-action, a unit map $\eta : 1 \to \mathcal{P}(1)$, an a collection of product maps, 
\[ \gamma : \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \to \mathcal{P}(n), \]
where $n = n_1 + \cdots + n_k$. These product maps are required to be unital, associative, and equivariant. A map $f : \mathcal{P} \to \mathcal{Q}$ between operads is a series $f_j : \mathcal{P}(j) \to \mathcal{Q}(j)$ of maps in $\mathcal{C}$ which commute with the product and unit maps.

In the product maps above, we should think of the $\mathcal{P}(k)$ and the $\mathcal{P}(n_i)$'s as two separate pieces of data. The $\mathcal{P}(k)$ piece somehow uses the $k$-many other random pieces to produce a larger piece.

Notice that although we demand the product operation of operads to be associative in some sense, the objects that operads model need not be associative. Just as the fact we are working in a symmetric monoidal category does not constrain our operads to model only commutative objects.

It is maybe unclear what is meant by these conditions the product maps have to satisfy (even the unital condition), so let us draw some diagrams. The unital condition is precisely the following two diagrams,

\[
\begin{align*}
1 \otimes \mathcal{P}(j) & \xrightarrow{\eta \otimes \text{id}} \mathcal{P}(j) \\
\mathcal{P}(1) \otimes \mathcal{P}(j) & \xrightarrow{\gamma} \mathcal{P}(j)
\end{align*}
\]

(Unital)

The horizontal maps in both diagrams are the canonical unit maps inside $\mathcal{C}$. This condition states the unit map $\eta$ has not effect on the product map $\gamma$. Next is the associativity condition.

\[
\begin{align*}
\mathcal{P}(k) \otimes \left( \bigotimes_{i=1}^{k} \mathcal{P}(n_i) \right) \otimes \left( \bigotimes_{i=1}^{n} \mathcal{P}(m_i) \right) & \xrightarrow{\gamma \otimes \text{id}} \mathcal{P}(n) \otimes \left( \bigotimes_{i=1}^{n} \mathcal{P}(m_i) \right) \\
\text{shuffle} & \downarrow \quad \downarrow \gamma \\
\mathcal{P}(k) \otimes \left( \bigotimes_{i=1}^{k} \left( \mathcal{P}(n_i) \otimes \bigotimes_{p=1}^{n_i} \mathcal{P}(m_{n_i + \cdots + n_{i-1} + p}) \right) \right) & \xrightarrow{\gamma} \mathcal{P}(m)
\end{align*}
\]

(Associative)

In the above diagram we have use the following abbreviations,

\[ n = \sum_{i=1}^{k} n_i, \quad m = \sum_{l=1}^{n} m_l, \quad q_i = \sum_{p=1}^{n_i} m_{n_1 + \cdots + n_{i-1} + p}. \]

This associativity condition says that when we stick two layers of trees together, it doesn’t matter whether we stick the middle layer onto the bottom, followed by the top layer into the middle, or if we slowly try and stick the top layer on the middle layer first, with care, and then stick what is left onto the bottom later. Draw a tree diagram if you wish.

Given $\sigma \in \Sigma_k$, and $\tau_i \in \Sigma_{n_i}$, we can define the sum permutation $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_n$, which is simple concatenation, and the block permutation $\sigma(n_1, \ldots, n_k) \in \Sigma_n$ which permutes $k$-many blocks of letters (of size $n_1, \ldots, n_k$) as $\sigma$ permutes $k$-many letters. Our equivariant condition on $\gamma$ is then stated in
regards to \( \sigma \), and separately with respect to a collection of \( \tau_i \)'s.

\[
\begin{array}{ccc}
\mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{P}(n) & \xrightarrow{\sigma(n_1,\ldots,n_k)} & \mathcal{P}(n)
\end{array}
\qquad \text{(Equivariance in } \sigma)\]

\[
\begin{array}{ccc}
\mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) & \xrightarrow{id \otimes \tau_1 \otimes \cdots \otimes \tau_k} & \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{P}(n) & \xrightarrow{\tau_1 \otimes \cdots \otimes \tau_k} & \mathcal{P}(n)
\end{array}
\qquad \text{(Equivariance in } \tau_i \text{’s)\]

These equivariant conditions simply say that if we jumble up the inputs into \( \mathcal{P}(k) \), then the outputs are going to be comparatively jumbled up.

We should think about operads as sets of operations, so each level \( \mathcal{P}(j) \) is a set of \( j \)-ary operations. These are operations which receive \( j \)-inputs, and produce one output. We can then clearly compose \( k \)-many operations of arbitrary arity into a \( k \)-ary operation. This is exactly what the product (sometimes called the composition) map does. There are many ‘prototypical’ examples of operads, or objects that were naturally operads before the term was even invented, but the example to keep in mind is the endomorphism operad.

**Example 1.2 (Endomorphism Operad).** Let \( A \) be an objects of our closed symmetric monoidal category \( \mathcal{C} \). Define an operad \( \text{End}_A \) to have \( j \)th level,

\[
\text{End}_A(j) = \text{hom}(A^j, A).
\]

The right \( \Sigma_j \)-action on \( \text{hom}(A^j, A) \) is simply the action permuting the \( A^j \)-factors, the unit map sends 1 to the identity map \( id : A \to A \) inside \( \text{End}_A(1) \), and the product map is simply,

\[
\text{End}_A(k) \otimes \text{End}_A(n_1) \otimes \cdots \otimes \text{End}_A(n_k) \to \text{End}_A(n)
\]

\[
f \otimes g_1 \otimes \cdots \otimes g_k \mapsto f \circ (g_1 \otimes \cdots \otimes g_k).
\]

It is said that group theory is boring without representations, so the analogy follows that operad theory is boring with its own representations in \( \mathcal{C} \), called algebras.

**Definition 1.3 (Algebra of an Operad).** Let \( \mathcal{P} \) be an operad inside \( \mathcal{C} \), then an algebra \( A \) over \( \mathcal{P} \) is an object \( A \in \mathcal{C} \), and a series of maps,

\[
\theta : \mathcal{P}(j) \otimes A^j \to A,
\]

for all \( j \geq 0 \) which are unital, associative and equivariant.

Rather than state this conditions again in detail, let us make the following observation. The maps \( \theta : \mathcal{P}(j) \otimes A^j \to A \) have adjoints \( \hat{\theta} : \mathcal{P}(j) \to \text{hom}(A^j, A) = \text{End}_A(j) \), which are also suitably unital, associative and equivariant. It turns out these conditions placed on the maps \( \theta \) are equivalent to the conditions placed on \( \Theta : \mathcal{P} \to \text{End}_A \) as a map of operads. Hence we define the conditions on \( \theta \) through this convention, and notice that any \( \mathcal{P} \)-algebra \( A \) is equivalent to a map of operads \( \mathcal{P} \to \text{End}_A \).

This is not how we should really think about algebras though, we should think about them as objects which feed inputs into our operads. For example the notation is suggestive, the maps,

\[
\theta : \mathcal{P}(j) \otimes A^j \to A,
\]

can be viewed in a concrete category\(^1\) as taking an element \( a_1 \otimes \cdots \otimes a_j \in A^j \) to \( \phi(a_1, \ldots, a_j) \), simply a \( j \)-ary operation \( \phi \in \mathcal{P}(j) \) evaluated on these elements. Then the unital condition implies that evaluating

\(^1\)I am not sure what the definition here is. Maybe a category with a forgetful functor to the category of sets? It is just a category where each object \( A \in \mathcal{C} \) has elements \( x \in A \) upon which we can define maps, at least set theoretically.
an element \(a \in A\) on the identity map in \(\mathcal{P}(1)\) is just the identity. The associativity of \(\theta\) is simply saying that evaluating \(a_1 \otimes \cdots \otimes a_j\) on a single \(\phi \in \mathcal{P}(j)\), where \(\mathcal{P}(j)\) comes from the product map \(\gamma_i\), is the same as evaluating each piece first, and then sticking together the operad using the product. Equivariance just says something about permuting the inputs of \(\phi \in \mathcal{P}(j)\) and the literal inputs \(a_1 \otimes \cdots \otimes a_j\) at the same time.

We will soon see some examples of operads and what algebras over these operads look like. We can continue down this algebraically inspired path though, and define what it means to be a module over an algebra, itself over an operad.

**Definition 1.4** (Module over \(A\)). Given an algebra \(A\) over an operad \(\mathcal{P}\) inside \(\mathcal{C}\), then a module over \(A\) is an object \(M \in \mathcal{C}\), equipped with structure maps, 
\[
\lambda : \mathcal{P}(j) \otimes A^{j-1} \otimes M \to M,
\]
for all \(j \geq 1\) which are suitably unital, associative and equivariant.

For \(j = 2\) we obtain our usual structure of an \(A\)-module, if \(A\) is an algebra over a monoid, a map \(\lambda : \mathcal{P}(2) \otimes A \otimes M \to M\), where \(\mathcal{P}(2)\) somehow has a little control over the module action. The module action for \(j \geq 3\) covers all higher coherances.

One other thing we will discuss briefly is the a partially defined product on an operad \(\mathcal{P}\).

**Definition 1.5.** Given an operad \(\mathcal{P}\), then define the partial product map \(\circ_i\) for \(1 \leq i \leq j\) as the following composite,
\[
\mathcal{P}(j) \otimes \mathcal{P}(k) \cong \mathcal{P}(j) \otimes 1 \otimes \cdots \otimes 1 \otimes \mathcal{P}(k) \otimes 1 \otimes \cdots \otimes 1 \\
\xrightarrow{\star} \mathcal{P}(j) \otimes \mathcal{P}(1) \otimes \cdots \otimes \mathcal{P}(1) \otimes \mathcal{P}(k) \otimes \mathcal{P}(1) \otimes \cdots \otimes \mathcal{P}(1) \xrightarrow{\gamma} \mathcal{P}(j+k-1),
\]
where we have inserted \((j-1)\)-many \(1\)'s with \(\mathcal{P}(k)\) is in the \(i\)th slot, and \(\star\) is the map,
\[
\star = id \otimes \eta \otimes \cdots \otimes \eta \otimes id \otimes \eta \otimes \cdots \otimes \eta.
\]
The is maybe not so obvious combinatorially why these partially defined maps \(\circ_i\) actually defined the whole product structure \(\gamma\) on our operad \(\mathcal{P}\), but it should be morally obvious. Each \(\circ_i\) simply isolates one particular component of \(\gamma\), hence they determine each other. Sometimes operads come to us only partially defined in a natural way, sometimes it is simpler to describe these partial product maps than \(\gamma\), and occasionally we have an object that is trying to be an operad and only has a handful of the partial product maps, perhaps not enough to constitute a whole operad, but we can study this operad-like object anyway.

These are all abstract definitions as we mathematicians like, but we also demand some purpose and interest, so now we will see some examples, followed by some applications, and then a generalisation.

## 2 Fundamental Examples

There are a handful of canonical and historical examples that we have to deal with now.

**Example 2.1** (Tree Operad). We can get into technicalities here, but we will resist and provide an example that is purely diagramatic, the tree operad, Tree, in the category of planar rooted trees with labeled leaves. Loosely speaking \(\text{Tree}(j)\) is simply a tree with \(j\)-ordered leaves, where \(\Sigma_j\) acts by permuting the leaves. The object \(\text{Tree}(1)\) is simply a tree with no branches, and \(\text{Tree}(0)\) the empty tree. The product of \(\text{Tree}(k)\) with \(\text{Tree}(n_1) \sqcup \cdots \sqcup \text{Tree}(n_k)\) simply places each \(\text{Tree}(n_i)\) onto the \(i\)th branch of \(\text{Tree}(k)\). This is how one can easily visually depict operads.
Example 2.2 (Associativity Operad). This is a quintessential operad to understand. First, recall that a monoid in a monoidal category is an object $A \in C$ with a unit $\eta : 1 \to A$ and product $\mu : A \otimes A \to A$, which are adequately unital and associative, and a left-module over a monoid $A$ is an object $M \in C$ and a sufficiently coherent action map $A \otimes M \to M$, and a bimodule is defined similarly.

Let $C$ be our closed symmetric monoidal category, then we define $\Ass$ by $\Ass(j) = 1[\Sigma_j]$, with right $\Sigma_j$-action simply by left translation, so $1[\tau] \sigma = [\tau \sigma]$. By definition $\Ass(0) = 1$ and the unit map is simply the identity $1 \to \Ass(1) = 1$. The product map is dictated by the strong equivariance (free and transitive) $\Sigma_j$-actions on $\Ass(j)$, but if we are allowed to use the notation $\sigma$ for the coproduct summand of $1[\Sigma_j]$ indexed by $\sigma$, for $\sigma \in \Sigma_j$, then we can express the product map $\gamma : \Ass(k) \otimes \Ass(n_1) \otimes \cdots \otimes \Ass(n_k) \to \Ass(n)$, as the following formula,

$$
\gamma(\sigma \otimes \tau_1 \otimes \cdots \otimes \tau_k) = (\tau_1 \oplus \cdots \oplus \tau_k) \cdot \sigma(n_1, \ldots, n_k).
$$

Let $A \in C$ be an algebra over this operad $\Ass$, then we have maps $\theta : \Ass(j) \otimes A^j \to A$. Let us consider these maps for some small values of $j$.

$(j = 0)$ Here we have a map $1 \to A$ which looks like a unit map if $A$ was a monoid in $C$, so we need a product map.

$(j = 1)$ The unital condition of $\theta$ is the following diagram,

$\begin{array}{ccc}
1 \otimes A & \xrightarrow{\approx} & A \\
\downarrow_{\text{id} = \eta \circ \text{id}} & & \downarrow_{\theta}
\end{array}$

which implies that $\theta : \Ass(1) \otimes A \to A$ is simply the canonical isomorphism $1 \otimes A \cong A$. Still no interesting mathematics.

$(j = 2)$ Now we obtain something useful, $\theta : \Ass(2) \otimes A \otimes A \to A$. The domain is just $1[\Sigma_2] \otimes A \otimes A$ which by our distributivity properties of is simply $1[\Sigma](A \otimes A) \sqcup 1[\sigma](A \otimes A)$ where $\sigma$ is the non-trivial element of $\Sigma_2$. Our map $\theta$ is then made of two maps $\theta_e : A^2 \to A$ and $\theta_\sigma : A^2 \to A$ subject to an equivariant relation. If we work inside a concrete category we can obtain a formula like,

$$
\theta_\sigma(a_1 \otimes a_2) = \theta_e(a_2 \otimes a_1),
$$

which simply tells us that $\theta_\sigma$ is determined by $\theta_e$, so we simply obtain a ‘reduced’ map $A \otimes A \to A$, which will be our product map $\mu : A \otimes A \to A$.

$(j \geq 3)$ For all higher $\theta : \Ass(j) \otimes A^j \to A$ our equivariance conditions gives us,

$$
\theta(\sigma \otimes a_1 \otimes \cdots \otimes a_k) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}, 
\text{ (2.3)}
$$

where $\sigma \in \Sigma_j$, which encodes all of the iterates and permutations of the product on a monoid. The unitality and associativity of $\theta$ then correspond to the unitality and associativity of $\mu : A \otimes A \to A$.

From this we see that $\Ass$-algebras are simply monoids inside $C$. The converse is also true, given a monoid $A$ inside $C$, we define $\theta$ using Equation 2.3 and the product $\mu$ on $A$, and again the unital and associative conditions on $\theta$ come from the corresponding conditions on $\mu$.

Given an $\Ass$-algebra $A$, then $A$-modules in the operad theory sense give off a particularly satisfying description as well. Let $M$ be a module over an $\Ass$-algebra $A$, then we have maps,

$$
\lambda : \Ass(j) \otimes A^{j-1} \otimes M \to M.
$$
We claim that an $A$-module $M$ is exactly an $A$-bimodule, where $A$ is now considered to be a monoid inside $C$. To see this we will construct left and right actions of $A$ on $M$, and from these left and right actions, recover the maps $\lambda$. Given an $A$-module in the operad theory sense, then we have action maps,

$$am = \lambda(c \otimes a \otimes m) \quad ma = \lambda(\sigma \otimes a \otimes m),$$

where $e$ and $\sigma$ are the two distinct elements of $\Sigma_2$. Conversely, if $M$ is an $A$-bimodule in the monoidal category theory sense, then we can define $A$-algebra operadic action maps,

$$\lambda(\sigma \otimes a_1 \otimes \cdots \otimes a_k) = a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

where $\sigma \in \Sigma_k, a_i \in A$ for $1 \leq i < k$ and $a_k \in M$.

**Example 2.4 (Commutivity Operad).** Slightly more trivial that the $Ass$ operad inside $C$, is the $Com$ operad inside $C$. Define $Com$ by setting $Com(j) = 1$ for all $j \geq 0$, with trivial $\Sigma_j$-action, unit map the identity $\eta : 1 \to Com(1) = 1$, and product map the canonical map,

$$\gamma : Com(k) \otimes Com(n_1) \otimes \cdots \otimes Com(n_k) = 1 \otimes 1 \otimes \cdots \otimes 1 \cong 1 \overset{id}{\to} 1 = Com(n).$$

All the conditions on $\gamma$ are fulfilled vacuously. A $Com$-algebra $A \in C$ comes with maps $\theta : Com(j) \otimes A^j \to A$, which are now simply maps $\theta : A^j \to A$. Since each $Com(j)$ is boring, one might be lulled into think our standard list of conditions on $\theta$ are trivial but this is not true! The associativity condition is now the following diagram,

$$\begin{array}{ccc}
A^n & \xrightarrow{\theta_n} & A \\
\downarrow & & \downarrow \\
A^{n_1} \otimes \cdots \otimes A^{n_k} & \xrightarrow{\theta_{n_1} \otimes \cdots \otimes \theta_{n_k}} & A^k \\
& \downarrow & \\
A \otimes \cdots \otimes A & \xrightarrow{\theta} & A^n
\end{array},$$

where we’ve given the $\theta$ maps little subscripts to clarify which one we are using. This diagram implies that all our higher order $\theta$’s come from smaller ones, so we really only need $\theta : A^2 \to A$, which is now literally our product map $\mu$. If we review the equivariance condition on $\theta : A^2 \to A$, we notice that this product must be commutative, given by the following diagram,

$$\begin{array}{ccc}
A^2 & \xrightarrow{\sigma} & A^2 \\
\downarrow & \xrightarrow{\theta} & \downarrow \\
A & \xrightarrow{\theta} & A
\end{array},$$

where the $\sigma \in \Sigma_2$ swaps our two $A$ factors. Notice the difference to the $Ass$-algebra situation, where we couldn’t derive such strong information from the equivariance conditions. This type of argument shows us $Com$-algebras are simply commutative monoids, and to see they are exactly commutative monoids in $C$ we need only trace through this argument backwards.

Now we move onto modules, so let $A$ be a $Com$-algebra and $M$ an $A$-module, then we have a collection of maps,

$$\lambda : Com(j) \otimes A^{j-1} \otimes M \cong A^{j-1} \otimes M \to M.$$

Once again, by the associativity condition, we see the higher $\lambda$ maps are all generated by lower degree $\lambda$ maps, so we need only concern ourselves with $\lambda : A \otimes M \to M$, but this precisely recognises $M$ as a left $A$-module. Notice, that in the modules over an $Ass$-algebra case, we constructed a left and a right module action using the fact that we have two copies of $1$ in $Ass(2)$ (corresponding to the two elements of $\Sigma_2$), but here we only have one $1$. This means that from a module $M$ over a $Com$-algebra $A$, we can only give $M$ the structure of a left $A$-module. Conversely, we take a left $A$-module $M$ and notice that our left module action map gives us $\lambda : A \otimes M \to M$, which by iterations and the product on $A$ lifts to higher $\lambda$’s.
We now take a stark shift away from general algebraic examples, and have a look at some topological operads, which helped utilise the power of operads in the early days.

3 Operads in Topology

Before operads were invented, the concept behind an operad was already driving some mathematical research. Jim Stasheff invented a series of polytopes called “associahedrons” which would eventually turn out to be a good example of an operad. Before we talk about this, let us remind ourselves about some stable homotopy theory as motivation.

Definition 3.1 (Sequential Spectrum). A sequential spectrum is a series of spaces $X_n$ for each $n \geq 0$, and a collection of structure maps $\sigma : \Sigma X_n \rightarrow X_{n+1}$ which are transitive, where $\Sigma Y = S^1 \wedge Y$ is the reduced suspension of a space $Y$.

We then define the homotopy groups of a spectrum as the colimit,

$$\pi_n(X) = \colim_k \pi_{n+k}(X_k).$$

If we define the suspension spectrum of a space $X$ as $\Sigma^\infty X$ with levels $(\Sigma^\infty X)_n = \Sigma^n X$. Using Freudenthal’s suspension theorem, the homotopy groups of $X$ stabilise and we obtain the stable homotopy groups of the space $X$,

$$\pi_n(\Sigma^\infty X) \cong \pi_n^s(X).$$

Notice that the structure maps of a spectrum $X$, $\sigma : \Sigma X_n \rightarrow X_{n+1}$ have adjoint maps $\tilde{\sigma} : X_n \rightarrow \Omega X_{n+1}$. Hence a sequential spectrum can be defined in terms of these adjoint structure maps. If these adjoint structure maps are (classically) a homeomorphism, or (contemporarily) a weak equivalence, then the homotopy groups of a spectrum are vastly simplified, and can be of a huge amount of interest. For example, the spectra which represent real and complex K-theory are built using these adjoint structure maps, hence it suffices to understand the spaces $BO$ and $BU$ in the zeroth level, rather than the whole spectra $KO$ and $KU$. These spectra also naturally carry periodicity phenomena, which have driven research in algebraic topology from Adams and Atiyah’s proof of the Hopf invariant 1 theorem, to more recently in Hill, Hopkins and Ravenel’s proof of the Kervaire invariant 1 theorem.

In short, we have a lot of interest in studying spaces like $BO$ and $BU$, which have maps $BO \rightarrow \Omega^n X$ for each $n \geq 0$ and some space $X$, which are weak homotopy equivalences. These types of spaces are called infinite loop spaces, and loop spaces have long interested topologists.

Definition 3.2 (Loop Space). A space $X$ is called a loop space if it is equal to $\Omega Y$ for some other space $Y$. A space $X$ is called an $n$-fold loop space if $X = \Omega^n Y$.

There are other reasons why mathematicians are interested in loop spaces too. For example, the points in an single loop space $\Omega X$ almost have a group structure. Given points $x$ and $y$ inside $\Omega X$, which are really pointed maps $\phi, \psi : S^1 \rightarrow X$, then we can add two loops the same way we add two elements of the fundamental group of $X$ together. In the fundamental group of $X$ we obtain a strictly associative group structure, since the fundamental group is interested in loops only up to homotopy. In the loop space $\Omega X$, the ‘group’ product is not associative, only associative up to homotopy,

$$\phi \ast (\psi \ast \theta) \simeq (\phi \ast \psi) \ast \theta.$$

We can symbolise these two orders of multiplying two loops as a tree, and their exists a homotopy $h_3 : I \times (\Omega X)^3 \rightarrow \Omega X$. When we want to compose four loops we have five possible orders, so five binary trees, and we can use our $h_3$ to move between certain trees. Their are two ways to go from $((\phi \ast \psi) \ast \theta) \ast \epsilon$ to $\phi \ast ((\psi \ast \theta) \ast \epsilon)$ using a series of $h_3$’s, and these two choices are homotopic by a homotopy $h_4 : K_4 \times (\Omega X)^4 \rightarrow \Omega X$, where $K_4$ is a solid pentagon bounded by two paths. Once we get
to 5 factors we have 14 choices and using \( h_3 \) and \( h_4 \) we obtain a new homotopy \( h_5 : K_5 \times (\Omega X)^5 \to X \), where \( K_5 \) is now becoming harder to describe.

In general we have maps \( h_j : K_j \times (\Omega X)^j \to \Omega X \) which satisfy some compatibility conditions. If these \( K_j \)'s formed an operad, then the above construction sees loops spaces as \( K \)-algebras (also called \( K \)-space).

**Example 3.3** (Associahedron Operad). Let \( K \) be Stasheff’s associahedron operad, defined by level-wise by letting \( K_j \) be the associahedrons partially defined above. Not only are loops spaces \( K \)-spaces, but in fact the converse is true in a homotopical sense.

**Theorem 3.4** (Stasheff). A \( K \)-space has exactly the homotopy type of a loop space.

Stasheff proved the above theorem using his associahedron’s before operads were invented. After the invention of operads, a generalisation of Stasheff’s result was available.

**Example 3.5** (Little Cubes Operad). Now we are working in the category of based topological spaces, with the cartesian product as the symmetric monoidal product, with identity the one-point space \(*\). Inside \( I^n \), where \( I \) is the unit interval, we define a little \( n \)-cube to be the image of an affine embedding \( I^n \to I^n \), where the image of the axes are parallel to the axes of the codomain. The \( n \)th little cubes operad, \( LC_n \), is defined by letting \( LC_n(j) \) be the space of \( j \)-many ordered little \( n \)-cubes, which are pairwise disjoint except possibly at the boundary. Let us define the partial product maps \( c_i \), which takes \( k \)-many little \( n \)-cubes and scales them into the \( i \)th little \( n \)-cube of \( LC_n(j) \), for \( 1 \leq i \leq j \). This is very nice to see visually.

There is also a definition of \( LC_\infty \), which is defined as the colimit of the finite \( LC_n \) (in the category of operads), along the inclusion morphisms of operads \( LC_n \to LC_{n+1} \) given by sending the little \( n \)-cube \( c \) to the little \((n + 1)\)-cube \( c \times \text{id}_I \).

Soon we will see that algebras over \( LC_n \), which maybe called \( LC_n \)-spaces, can be characterised in a particularly interesting (and perhaps predictable) way up to homotopy.

Let \( X = \Omega^n Y \), then we will show that \( X \) is a \( LC_n \)-space, using the action map originally constructed by Boardman and Vogt,

\[
A_{BV} : LC_n \to \text{End}_X.
\]

This map sends the \( j \)-many little \( n \)-cubes \( c = (c_1, \ldots, c_j) \) to the endomorphism of \( X \), which sends the \( n \)-tuple \((\beta_1, \ldots, \beta_j)\), with \( \beta_i \in \Omega^n Y \), so \( \beta : (I^n, \partial I^n) \to (Y, \ast) \), to the element of \( \Omega^n Y \) which applies \( \beta_i \) in the image of \( c_i \), and collapses everything outside these \( c_i \)'s, giving us a map \((I^n, \partial I^n) \to (Y, \ast)\). We of course should check the unital, associative and equivariance properties of \( A_{BV} \), but this has been done.

It is in this way that we can see \( k \)-fold loops spaces as \( LC_k \)-spaces, and the converse is also true up to homotopy type.

**Theorem 3.6** (Boardman, May, Vogt). A connected \( LC_k \)-space \( X \) is of the homotopy type of a \( k \)-fold loop space.

**Sketch of May’s Result.** May considered the two sided bar construction, \( B(L, A, R) \). A chain of weak equivalences was then constructed for a \( LC_n \)-space \( X \),

\[
X \leftarrow B(LC_n, LC_n, X) \to B(\Omega^n \Sigma^n, C_n, X) \to \Omega^n B(\Sigma^n, LC_n, X),
\]

which directly shows \( X \) has the homotopy type of an \( n \)-fold loop space.

In fact, the result above does not strictly depend on the operad \( LC \), in fact the same statement can be made about a more general family of operads.
Definition 3.7 (\(A_\infty\)-operad). An operad \(\mathcal{P}\) in a model category (usually of differentially graded \(k\)-modules for a field \(k\), spaces or spectra) is said to be an \(A_\infty\)-operad if each \(\mathcal{P}_j\) is contractible.

We think of \(A_\infty\)-operads as having some sort of strong associativity properties up to homotopy.

Definition 3.8 (\(E_\infty\)-operad). An operad \(\mathcal{P}\) is a model category is said to be an \(E_\infty\)-operad if \(\mathcal{P}\) is an \(A_\infty\) operad, and each \(\mathcal{P}_j\) has a free \(\Sigma_j\)-action.

These \(E_\infty\)-operads have some type of strong commutivity properties, again up to homotopy. If we are working with topological spaces, we can view \(\mathcal{P}_k\) of an \(E_\infty\)-operad as the total space of a universal \(\Sigma_k\)-bundle. An infinite version of the theorem of Boardman, May and Vogt above reads as follows.

Theorem 3.9 (Boardman, May, Vogt). A connected space \(X\) has the homotopy type of an infinite loop space if and only if \(X\) is a \(\mathcal{P}\)-space for some \(E_\infty\)-operad \(\mathcal{P}\).

Using the Eckmann-Hilton argument, one can prove that the product action on \(k\)-fold loop spaces for \(k \geq 2\) are commutative, so it might not be surprising that once we leave the one dimensional case, we are interested in commutativity up to homotopy in addition to associativity. One example of an \(E_\infty\)-operad is the following, which comes up in many places, such as some foundations of the stable homotopy category, and the construction of an equivariantly connective real (or complex) cobordism spectrum.

Example 3.10 (Linear Isometries Operad). Let \(\text{hom}(V, \mathbb{R}^\infty)\) be the the space of linear isometric embeddings from the real inner-product space of countable dimension \(V\) to \(\mathbb{R}^\infty\), which is the colimits of \(\mathbb{R}^n\’s\), topologised by the compactly generated function space topology. We then define the linear isometries operad \(\mathcal{L}\) in the category of topological spaces by \(\mathcal{L}_j = \text{hom}(\bigoplus_j \mathbb{R}^\infty, \mathbb{R}^\infty)\). The \(\Sigma_j\)-action and product maps are given in the same way as the endomorphism operad, where the monoidal product in the category of real inner-product spaces of countable dimension is the direct sum.

It is a little theorem in topology that say \(\text{hom}(V, \mathbb{R}^\infty)\) is contractible for all allowable \(V\), and the \(\Sigma_j\)-action is obvious free, hence \(\mathcal{L}\) is an \(E_\infty\)-operad.

4 Operads in Algebra

Although we never really defined the associahedron operad \(\mathcal{K}\) properly, we can still use the theory to make some algebraic constructions. For example the spaces involved in the operad \(\mathcal{K}\) are CW-complexes, so we can take the cellular chain complex with \(k\)-coefficients associated to \(\mathcal{K}\), and then say a differentially graded \(k\)-module is an \(A_\infty\)-algebra if it is an algebra (in the operad theory sense) over this new ‘homotopy associative’ operad. We can then unpack this definition and make a more explicit definition, but we won’t do that here.

Instead, let us just note that these \(A_\infty\)-algebras are associative only up to homotopy, but their homology is strictly associative.

5 Coloured Operads

To set up the next seminar on \(\infty\)-operads, we need to introduce coloured operads, since \(\infty\)-operads are actually higher homotopical versions of coloured operads. Let us start with some observations.

Remark 5.1 (Observation). Let \(\mathcal{P}\) be an operad in a category \(\mathcal{C}\), then the datum of \(\mathcal{P}\) is equivalent the following:

1. An assignment \(\mathcal{P}_I\) for each finite set \(I\).
2. For each map \( I \to J \) of finite sets with fibres \( \{I_j\}_{j \in J} \) we have a map,
\[
\gamma : \prod_{j \in J} P_{I_j} \otimes P_J \to P_I.
\]

3. There exists elements \( \text{id} \in P_* \) which are the appropriate left and right units of \( \gamma \).

4. The maps \( \gamma \) are associative in same sense.

This is equivalent datum to our original definition of an operad, where the \( \Sigma_j \)-actions come from these \( \gamma \) maps induced by permutation automorphisms of the set with \( j \)-many letters.

A coloured operad is going to be defined in a way very similar to the data we have given above, except we want to keep track of colours, or specific varieties of objects. For example, we can image an operad with colours red and blue, where we can only compose \( \phi \in P_J \) and \( \psi \in P_k \) using the partial product map \( \circ_i \) if the \( i \)th stalk of \( \psi \) is the same colour as the root of \( \phi \). Try to draw a picture of this, and compare with the following definition.

**Definition 5.2** (Coloured Operad). A coloured operad \( P \) is the following data:

0. A collection \( \{X, Y, Z, \ldots\} \) which we call the objects, or the colours of \( P \).

1. For each finite set \( I \), each \( I \)-index collection of objects of \( P \), \( \{X_i\}_{i \in I} \), and each object \( Y \) we have a set,
\[
\text{Mul}_P(\{X_i\}_{i \in I}, Y),
\]
which we call the morphisms from \( \{X_i\}_{i \in I} \) to \( Y \).

2. For each map of finite sets \( I \to J \) with fibres \( \{I_j\}_{j \in J} \), each finite collection of objects \( \{X_i\}_{i \in I} \), each finite collection of objects \( \{Y_j\}_{j \in J} \), and each object \( Z \) of \( P \), we have the following product map,
\[
\gamma : \prod_{j \in J} \text{Mul}_P(\{X_i\}_{i \in I_j} \times \text{Mul}_P(\{Y_j\}_{j \in J}, Z) \to \text{Mul}_P(\{X_i\}_{i \in I}, Z).
\]

3. Elements \( \text{id}_X \in \text{Mul}_P(\{X\}, X) \) for all objects \( X \) in \( P \) which are appropriate left and right units of \( \gamma \).

4. The maps \( \gamma \) are appropriately associative.

A regular operad is simply a coloured operad with only one colour \( \mathbb{1} \), and we called \( \text{Mul}(\mathbb{1}^J, \mathbb{1}) \) simply \( P_J \).

In higher algebra, Lurie constructs a category \( P^{\otimes} \) corresponding to a coloured operad \( P \), and (in much the same way as he proceeds with for symmetric monoidal categories) shows that \( P^{\otimes} \) and a forgetful functor \( \pi : P^{\otimes} \to \text{Fin}_* \) (where the later category is Segel’s category of finite pointed sets) are equivalent to the the operad \( P \), up to canonical equivalence.

It is then possible to define \( \infty \)-operads inspired by this alternative approach.

**References**

[1] Higher Algebra

[2] Peter May’s Notes

[3] Peter May’s Original Loop Spaces Book